RANDOM ZEEMAN INTERACTIONS

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Math. contributions by
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Methods based on F-Spencer
multi-scale perturbation th.

Sorry, no SUSY, here.

For my mentor, collaborator
and friend
Tom
to whom I owe so much.

1. Introduction

Condensed-matter th. rich source of interesting math. problems in quantum th., incl. ones involving

dissipation

(systems coupled to reservoirs: decoherence, friction, QBM, hysteresis, ...) and

disorder/noise

(Anderson localization – linear & NL, Mott trans., disordered magnets, RF-Ising model, spin glasses, ...)

Some major problems:

- crystals & quasi-crystals
- magnetism ... • BEC
- BCS • FQHE • Mott trans.
- transport phenomena ...

Today:

Anderson localization for

- Bernoulli random pot.
- random Zeeman int.

(multi-scale analysis)

A case study:

$$Eu_x\,Ca_{1-x}\,B_6 \qquad (0 \leq x \leq 1)$$

$\uparrow$ mag. $\qquad \uparrow$ non mag.

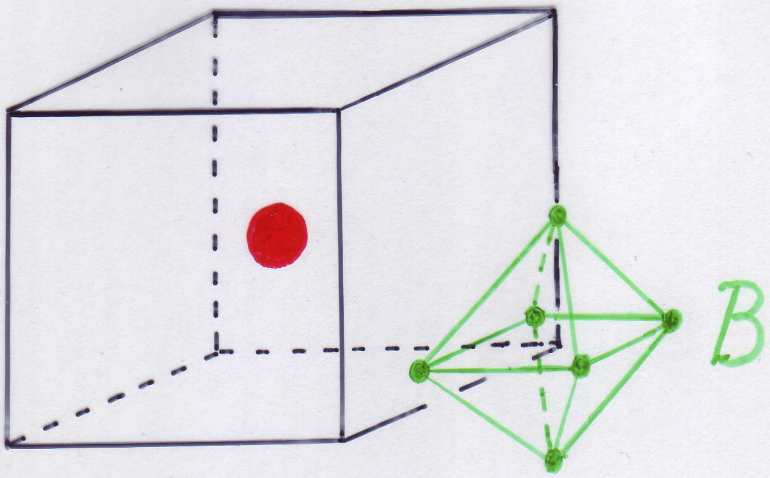
iso-electronic

- magn. ordering in coexist. with metallic props., for $x > x_c \simeq 0.3$ (percolation thr.)
- Mott trans., as x varies; localization from spin disorder
- Colossal negative magneto-resistance

2. Exp. & theor. results

Exp. Wigger ... Felder ... Ott ...
Bianchi ... Fisk ...

Structure of $Eu_x Ca_{1-x} B_6$:



$\bullet \longleftrightarrow$ Eu or Ca

$S = 7/2 \, (\frac{1}{2} \, 4f) \; | \quad 0$

Hund divalent

Lattice $\mathbb{Z}^3$; substitution of Eu by Ca is isoelectr.

Band structure (conj.)

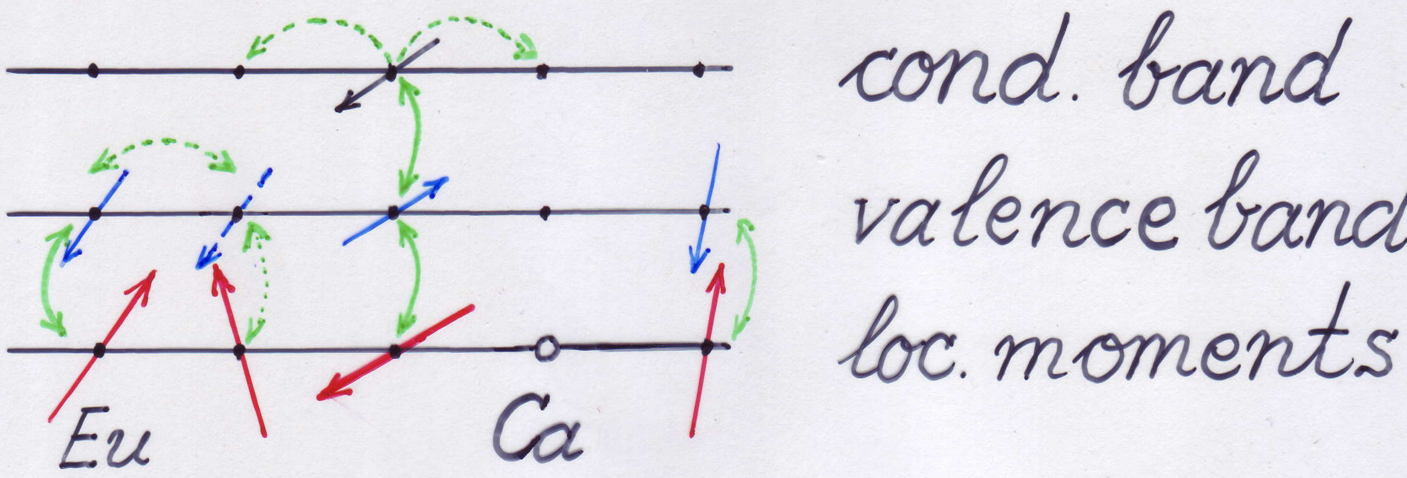
(l) ~ flat band of loc. magn. moments, $S = \frac{7}{2}$, (Eu), or $\emptyset$, (Ca).

(m) (~ 1/4 filled) valence band (Coulomb rep.)

(h) weakly filled cond. band (from small density of defects): $c_{el.} \simeq 10^{-4} \div 10^{-3}$ (defect- and T-dep.)

Material is a semi-conductor with "small gap".

Kuneš - Picket (PR B <u>69</u>, 04):

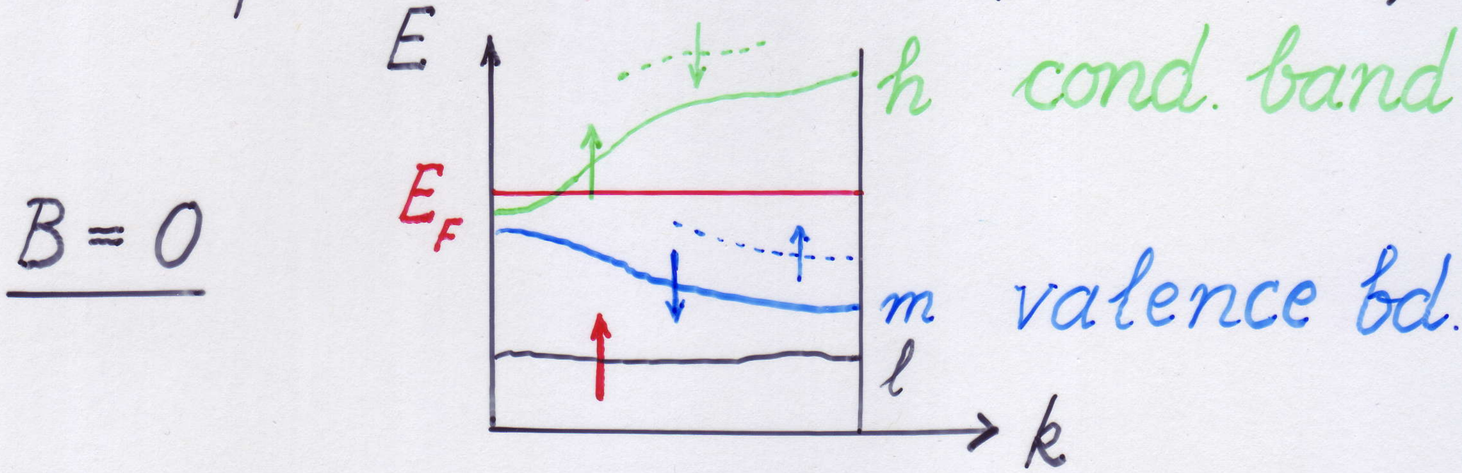


Zener, Anderson, ⋯ (F-Ue):

Ferromagn. ordering through indirect exchange, for $x > x_c$.

($T_c(x=1) \simeq 12.5 \div 15\,K$; $\nearrow C_{mag.}(T)$)

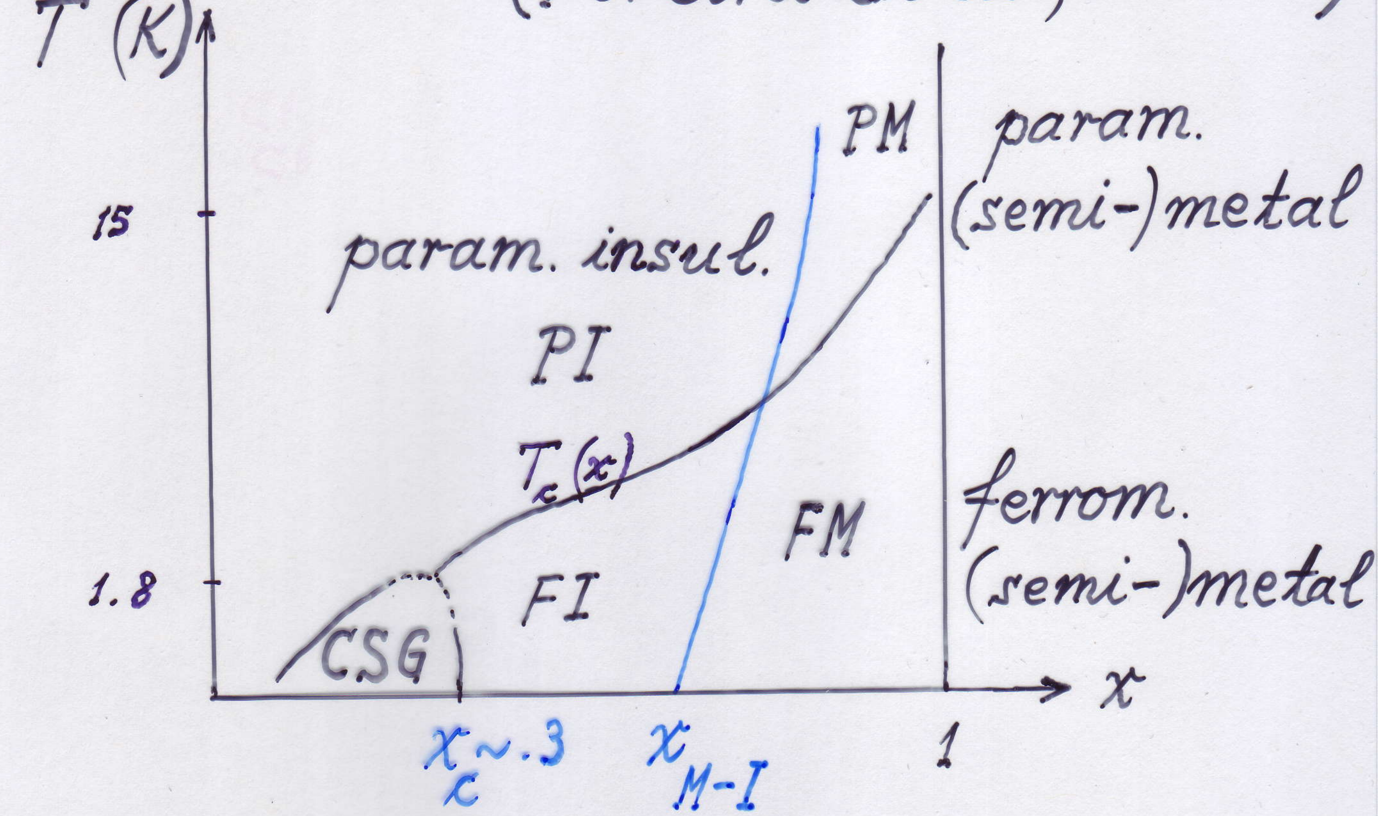
k-space band str. (idealized):



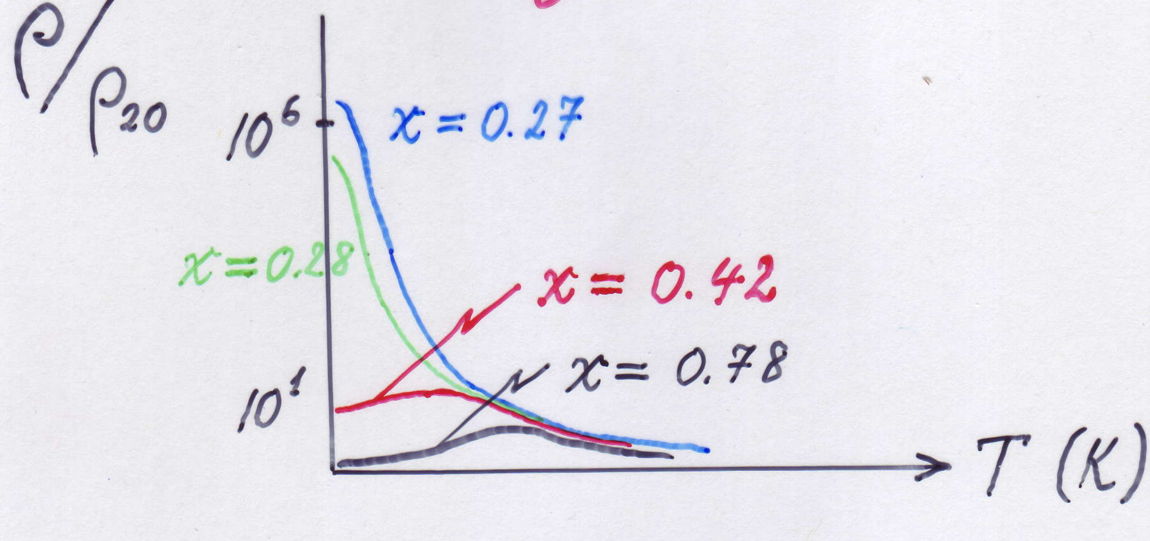
[For $0.12 < x < x_c$: "cluster spin glass"]

Phase diagram at B = 0

(Pereira et al., PRL '04)

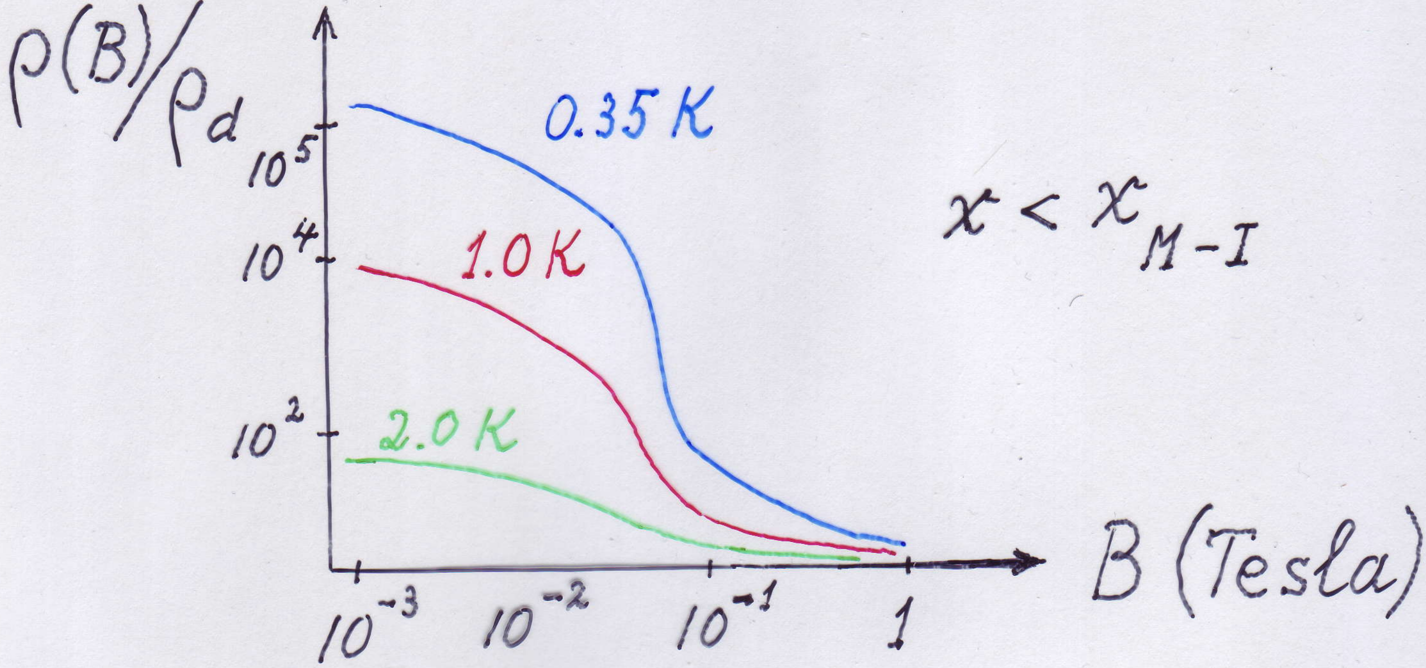


Resistivity



$$x_c < 0.42 < x_{M-I} < 0.78$$

Colossal negative MR



Origin of CnMR:

e^- carry magn. moment. Turn on $\vec{B} \Rightarrow$ magn. moments align w. $\vec{B} \Rightarrow \vec{M} \parallel \vec{B}$ in all Eu-clusters $\Rightarrow$ Magn. disorder felt by e^- in conduction band ↘ $\Rightarrow$ mobility edge, E_c, ↘ !

3. Metal-insulator (Mott) transition at $B = 0$

3.1 "Derivation" of effective Hamiltonian

Because e^- conc. in cond. band very small, neglect $e^- - e^-$ int.

→ One-particle Hamiltonian
+ Pauli principle

e^--spin in cond. band ferromagn. correlated with spins of loc. Eu-ions (↗ Sect. 2)

→ Kondo-lattice Hamiltonian

Born-Oppenheimer approx.:

Precession of aligned Eu-spins in conn. Eu-island slow compared to motion of e^- in cond. band $\rightarrow$ Treat Eu-spins as static.

Since tot. spin in conn. Eu-island is large, describe Eu-spins classically (Lieb).

$\rightarrow$ Eff. Hamiltonian of e^- in conduction band given by:

$$H(\omega) = T + V(\omega) + Z(\omega) \quad (1)$$

where

$$T \overset{e.g.}{=} -\Delta, \quad V(\omega) = \left(v_j(\omega)\right),$$

$$Z(\omega) = \left(\vec{h}_j(\omega)\cdot\vec{\sigma}\right), \qquad (2)$$

$j \in \mathbb{Z}^3$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

Distributions of $v_j(\omega)$, $\vec{h}_j(\omega)$:

$$v_j(\omega) = \begin{cases} v, & \text{w. prob. } x \\ -v, & \text{w. prob. } 1-x \end{cases}$$

$v \neq 0$ some small const.; Eu- and Ca-ions assumed to be put down according to site percolation process.

$$\vec{h}_j(\omega) = \begin{cases} \vec{h}, & j \in C_{Eu}^{conn.}, \\ 0, & j \in C_{Ca} \end{cases}$$

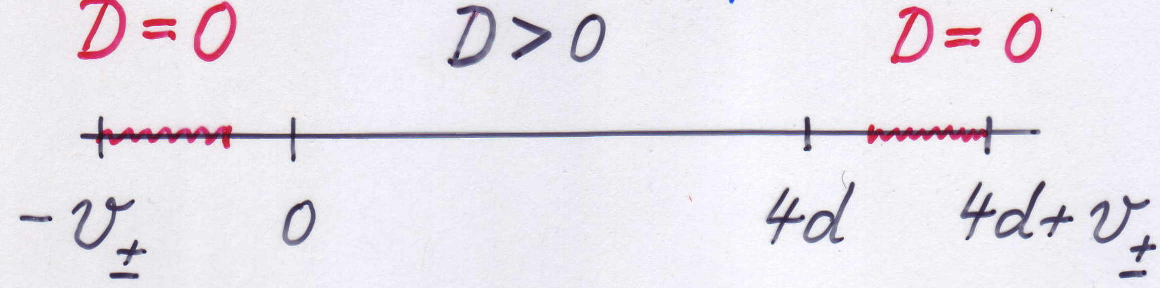
$|\vec{h}| = h = $ const.; direction of $\vec{h} = \vec{h}_{C_{Eu}^{conn}}$ only dep. on $C_{Eu}^{conn.}$

(i) If $x > x_c$ (ferromagn. ordering) $\exists$ ∞ conn. cluster, $\mathcal{C}^{\infty}_{Eu}$, and we approximate $H(\omega)$ by Bernoulli Hamiltonians for "spin-up" & "spin-down": $H^{(i)}_{\pm}$

(ii) If $x < x_c$: $\mathbb{E}\left|\mathcal{C}^{\text{conn.}}_{Eu}\right| < \infty$, directions of $\vec{h}_i(\omega)$ & $\vec{h}_j(\omega)$ uncorrelated if i & j belong to different Eu-clusters: $H^{(ii)}$

3.2 Conjectured spect. properties

(i)

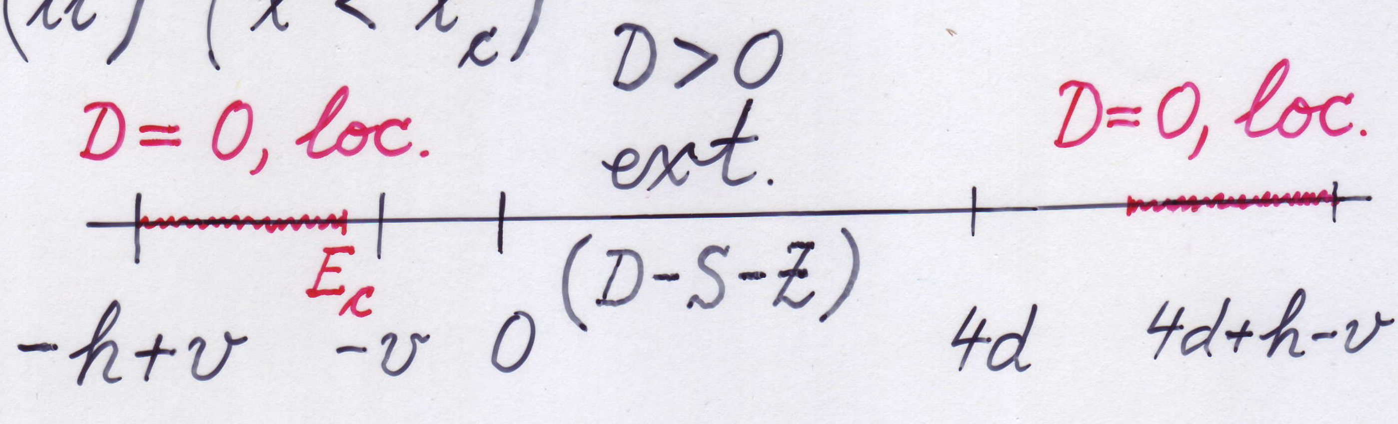


$$\rho(E)D(E) = \lim_{\varepsilon \searrow 0} \frac{2\varepsilon^2}{\pi d} \sum_{x\in\mathbb{Z}^d} |x|^2 \times$$

$$\times\ \mathbb{E}\left|\langle 0|(H(\omega)-E-i\varepsilon)^{-1}|x\rangle\right|^2$$

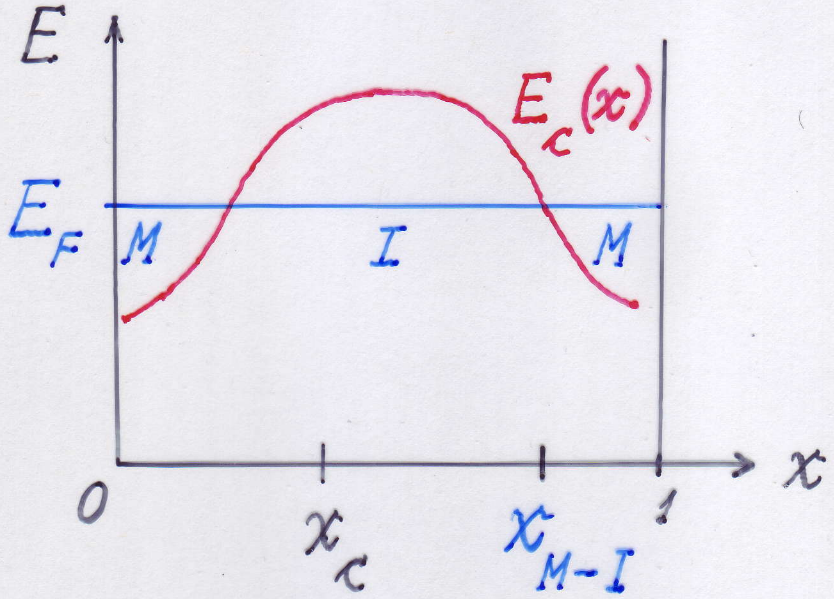
$$\sigma = \frac{e^2}{h}\int^{E_F} dE\, \rho(E)D(E).$$

(ii) $(x < x_c)$



Conjecture. For moderate amount of disorder, $\sigma(H(\omega))$ is ac in middle of band.

Application to M-I trans.



3.3 Results.

Theorem 1. ($\nearrow$ T. Spencer, 1993, ...)

$$H(\omega) = H_{\pm}^{(i)}(\omega), \quad v_j(\omega) = \begin{cases} v_{\pm}, & p \\ -v_{\pm}, & 1-p \end{cases}$$

$(p \simeq 1/2)$. Then $\rho(E) D(E) = 0$ on $[-|v_{\pm}|, -c|v_{\pm}|^2]$, exc. possibly on subset of measure $O\left(\exp\left(-\exp\frac{1}{\sqrt{|v_{\pm}|}}\right)\right)$, $|v_{\pm}|$ small.

Theorem 2. (Bourgain, Egli, F)

$$H(\omega) = H^{(ii)}(\omega), \; |\vec{h}_j(\omega)| = \begin{cases} h, & x \\ 0, & 1-x \end{cases}$$

(as descr. above). Let

$$I_\varepsilon := [-h, -h + \varepsilon(x,h)]$$

Then, for $0 < x < x_c$, $h > 0$ (small enough), $\varepsilon(x,h) > 0$ &

$$\sigma(H^{(ii)}(\omega)) \cap I_\varepsilon \subset \sigma_{pp}(H^{(ii)}(\omega)).$$

Corresp. eigenfunctions decay expon.

$$D(E) = 0, \; E \in I_\varepsilon.$$

Special case of general class of results!

3.4 Ideas underlying proof of Theorem 2.

Consider special case $T \overset{e.g.}{=} -\Delta$,

$v(\omega) \equiv 0$, $Z(\omega) = \left(\vec{h}_j(\omega)\cdot\vec{\sigma}\right)_{j\in\mathbb{Z}^3}$

$\vec{h}_j$'s i.i.d. r.v.'s with

$$dP(\vec{h}) = \left[x\, c_v\, e^{v\vec{e}_z\cdot\vec{h}}\,\delta\!\left(|\vec{h}|^2 - h^2\right) + (1-x)\,\delta^{(3)}(\vec{h})\right] d^3h$$

More general distr. for $\vec{h}_\cdot(\omega)$ viewed as "small" pert. of these (using cluster expansions).

Standard intuition into AL:

"T a "small" pert. of $Z(\omega)$".

But:
$$\sigma(\vec{h}_j \cdot \vec{\sigma}) = \{-h, 0, h\} : \infty$$

degenerate $\Rightarrow$

highly deg. pert. th.; no easy Wegner estimates!

Apply multi-scale analysis in conj. w. novel Wegner est.

A few details.

Cover $\mathbb{Z}^3$ w. ℓ-cubes, Λ_ℓ, sides of length $\ell = \ell_n$,

$$\ell_n \ll \ell_{n+1} \lesssim \ell_n^2, \quad n = 0, 1, 2, \cdots . \quad (3)$$

$$H_\Lambda := -\Delta_\Lambda + Z(\omega)\chi_\Lambda \qquad (4)$$

A cube $\Lambda = \Lambda_\ell$ "good" at energy

E if $G_\Lambda(E) := (H_\Lambda - E)^{-1}$ satisf.

$$\|G_\Lambda(E)\| \leq e^{\sqrt{\ell}}, \qquad (5)$$

$$|G_\Lambda(E; x, y)| < e^{-c|x-y|}, \qquad (6)$$

for $|x-y| > \frac{\ell}{10}$, $c > 0$.

Cube violating (5), (6) called "bad". Must prove (ind. in n) that prob., p_ℓ, that ℓ-cube Λ_ℓ is "good" is est. by

$$p_\ell \geq 1 - \text{const.}\, \ell^{-k}, \qquad (7)$$

for $E \in I_\varepsilon$, $\forall \ell = \ell_n$, $n = 0, 1, 2, \cdots$, with k large enough.

Prove (5) - (7) at some scale $\ell = \ell_0$ directly and then inductively, for $n = 1, 2, 3, \dots$.

(5) $\leftrightarrow$ Wegner-type est.: new

(6) $\leftrightarrow$ decay: multi-scale analysis

Induction step.

Assume (5) - (7) for $\ell = \ell_n$.

Consider some $\Lambda_{\ell_{n+1}}$. With high prob., $\Lambda_{\ell_{n+1}}$ contains only few "bad" ℓ_n-cubes $B^{(i)}_{\ell_n} \subset \Lambda_{\ell_{n+1}}$, $i = 1, 2, \dots$.

In a first step, we decouple

$B^{(i)}_{l_n}$, $i=1,2,\cdots$, from rest of $\Lambda_{l_{n+1}}$ (imposing Dirichlet b.c.)

$\Rightarrow (5)_{n+1}, (6)_{n+1}$ from $(5)_n, (6)_n$!

To make this precise, use 2^{nd} RId.: $X \subset \mathbb{Z}^3$, $\mathcal{Y} \subset X$. Then

$$G_X = G_{\mathcal{Y}} \oplus G_{X\setminus\mathcal{Y}} + (G_{\mathcal{Y}} \oplus G_{X\setminus\mathcal{Y}})\,\Gamma\, G_X \quad (8)$$

Γ: couplings betw. $\mathcal{Y}$ & $X\setminus\mathcal{Y}$.

For $x \in \mathcal{Y}$, $y \in X\setminus\mathcal{Y}$,

$$G_X(E;x,y) = \sum_{\langle z,z'\rangle \in \partial\mathcal{Y}\cap X} G_{\mathcal{Y}}(E;x,z) \times G_X(E;z',y) \quad (9)$$

If $B^{(i)}_{l_n}$'s are all decoupled $(5)_{n+1}, (6)_{n+1}$ follow from $(5)_n, (6)_n$

by using (9) in "Simon's cubes estimate", as in F-Sp.
Next, must couple $B_n^{(i)}$'s back to $\Lambda_{l_{n+1}}$ & show that $\Lambda_{l_{n+1}}$ is "good" with prob. $p_{l_{n+1}} \geq 1 - \text{const.} l_{n+1}^{-k}$
To accomplish this, cover $B_n^{(i)}$ by larger cube $\overline{B}_n^{(i)}$, with $\text{dist}(\partial \overline{B}_n^{(i)}, B_n^{(i)}) \geq c \cdot l_n$. Then modify config $\omega|_{B_n^{(i)}}$ to some $\omega'|_{B_n^{(i)}}$ s.t.

$$\| G_{B_n^{(i)}}(E, \omega') \| \leq e^{\sqrt{l_{n+1}}} \qquad (10)$$

The "2-contour lemma" of

F-Spencer enables us to couple $B_n^{(i)}$ back to $\Lambda_{\ell_{n+1}}$ in such a way that $(5)_{n+1}$, $(6)_{n+1}$ follow.

Next, we must show that prob. of $\{\omega'/_{B_n^{(i)}} \mid (10) \text{ holds}\}$, $\forall i = 1, 2, \cdots$, is so large that

$$\text{prob.}(\{\omega \mid \Lambda_{\ell_{n+1}} \text{“good”}\}) \equiv p_{\ell_{n+1}}$$
$$\geq 1 - \text{const.}\, \ell_{n+1}^{-k}.$$

Here the new ingredient (due to Bourgain) appears:

A matrix Cartan lemma!

<u>Lemma.</u> Let $M(\omega)$ be a real-analytic $N \times N$ matrix function of $\omega \in \Omega \overset{e.g.}{=} [a,b]^n$ satisfying:

(1) $M(\omega)$ has analytic ext. to $M(\xi)$, $\xi \in D^n$,

$D = \left\{ z \in \mathbb{C} \,\middle|\, \left| z - \frac{a+b}{2} \right| < \frac{e}{2} |a-b| \right\}$,

with $\|M(\xi)\| < B_1$, $\xi \in D^n$.

(2) There is $\Lambda \subset \{1, 2, \cdots, N\}$, w. $|\Lambda| \leq l \ll N$ s.t., $\forall \xi \in D^n$,

$$\left\| \left(M(\xi)_{\sim\Lambda} \right)^{-1} \right\| < B_2 .$$

(3) For some $\omega' \in \Omega$,

$$\| M(\omega')^{-1} \| < B_3 .$$

Then

$$\mu(\{\omega \in \Omega \mid \| M(\omega)^{-1} \| > K\})$$

$$< C \|m\|_\infty n |a-b|^n \times$$

$$\times K^{-(c/\ell \cdot \log B_1 B_2 B_3)}$$

($\mu = m \times$ Lebesgue measure).

Application.

$$M(\omega) = H_{B_n}(\omega) - E$$

Pf. of Lemma: Feshbach.

Proof that prob.$\{\exists$ "bad" l_0-cube$\}$ is small relies on simple intuition: ....